

(I₂). Energy of a Point charge distribution:

For charge q_1 , work done = 0, since there is no field present.

Bringing charge q_2 ,

work done will be $q_2 V_1(\vec{x}_2)$,

$V_1 \rightarrow$ Potential due

to q_1 .

$\vec{x}_2 \rightarrow$ denotes the position where we put charge q_2 .

Now $W_1 = 0$

$$W_2 = \frac{1}{4\pi\epsilon_0} q_2 \left(\frac{q_1}{|\vec{x}_{12}|} \right) \quad \text{--- (2)}$$

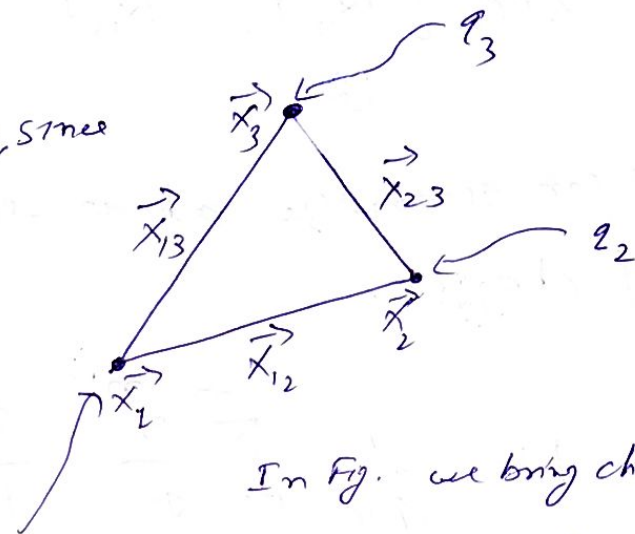
Similarly, for bringing charge q_3 , work done

$$W_3 = \frac{1}{4\pi\epsilon_0} q_3 \left(\frac{q_1}{|\vec{x}_{13}|} \right) + \frac{1}{4\pi\epsilon_0} q_3 \frac{q_2}{|\vec{x}_{23}|}$$

$$\text{or } W_3 = \frac{1}{4\pi\epsilon_0} q_3 \left[\frac{q_1}{|\vec{x}_{13}|} + \frac{q_2}{|\vec{x}_{23}|} \right] \quad \text{--- (3)}$$

Similarly for any charge q_4 , work done

$$W_4 = \frac{1}{4\pi\epsilon_0} q_4 \left[\frac{q_1}{|\vec{x}_{14}|} + \frac{q_2}{|\vec{x}_{24}|} + \frac{q_3}{|\vec{x}_{34}|} \right] \quad \text{--- (4)}$$



In Fig. we bring charges one by one from far away

Now the total work done to assemble the first four charges will be given by

$$W = W_1 + W_2 + W_3 + W_4$$

$$\text{or } W = \frac{1}{4\pi\epsilon_0} \left[\frac{q_1 q_2}{|\vec{x}_{12}|} + \frac{q_1 q_3}{|\vec{x}_{13}|} + \frac{q_1 q_4}{|\vec{x}_{14}|} + \frac{q_2 q_3}{|\vec{x}_{23}|} + \frac{q_2 q_4}{|\vec{x}_{24}|} + \frac{q_3 q_4}{|\vec{x}_{34}|} \right]$$

We can generalize above expression for n number of charges. ⑤

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \sum_{\substack{j=1 \\ j>i}}^n \frac{q_i q_j}{|\vec{x}_{ij}|}$$

} $j > i$ make sure that no pair is repeated

OR we can write above expression

$$W = \frac{1}{2} \left[\frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \frac{q_i q_j}{|\vec{x}_{ij}|} \right]$$

} $\frac{1}{2}$ factor is multiplied so that repeated pair does not affect the final result

$$W = \frac{1}{8\pi\epsilon_0} \sum_{\substack{i,j=1 \\ i \neq j}}^n \frac{q_i q_j}{|\vec{x}_{ij}|}$$

⑥

$$W = \frac{1}{2} \sum_{i=1}^n q_i \left[\sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{4\pi\epsilon_0} \frac{q_j}{|\vec{x}_{ij}|} \right]$$

⑦

Term in the square bracket represents the potential at point \vec{x}_i , due to all other charges q_j . Therefore, we

can write

$$W = \frac{1}{2} \sum_{i=1}^n q_i V(\vec{x}_i)$$

⑧

(Q2) Energy of a continuous charge distribution:

From (8), we see that for volume charge density ρ , this can be written as.

$$W = \frac{1}{2} \int \rho V d\tau \quad \text{--- (9)}$$

for surface charge and line charges, W is given by respectively. $\int \sigma V da$ and $\int \lambda V dl$.

The Gauss's law state that $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

$$\Rightarrow \rho = \epsilon_0 \nabla \cdot \vec{E} \quad \text{--- (10)}$$

Eq. (9) can be written as

$$W = \frac{1}{2} \int \epsilon_0 \nabla \cdot \vec{E} V d\tau = \frac{\epsilon_0}{2} \int (\nabla \cdot \vec{E}) V d\tau$$

Now using integration by parts in the above equation

$$W = \frac{\epsilon_0}{2} \left[- \int \vec{E} \cdot \nabla V d\tau + \oint V \vec{E} \cdot d\vec{a} \right]$$

since $\nabla V = -\vec{E}$, therefore we can write

$$W = \frac{\epsilon_0}{2} \left[\int_V E^2 d\tau + \oint_S V \vec{E} \cdot d\vec{a} \right]$$

Integrating over all space, the second term, surface integral, becomes zero. and we have

$$W = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 d\tau \quad \text{--- (11)}$$